Using Eq. (24), λ and r defined in Eqs. (21) and (22) can be expressed as functions of u, q, and t.

Conclusions

This Note presents a projective interpretation of the well-known Maggi's approach to dynamic analysis of nonholonomic systems. Both linear and nonlinear constraint cases were dealt within a unified treatment. The projection method used does not require the use of variational principle. Instead, the language of vector spaces is used, and tensor algebra analysis is applied to clarify the mathematical transformations and render the formulation compact.

Acknowledgments

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Probing Behavior in Certain Optimal Perturbation Control Laws

Warren W. Willman*
Naval Weapons Center, China Lake, California 93555

Introduction

THE results here are only formal in nature because they are based on optimal control law approximations that are derived with a dynamic programming analysis in which neglected error terms are assumed without proof to be sufficiently well behaved that all quantities that appear to be of negligible orders of magnitude actually are so in some precise and appropriate sense. Unless otherwise stated in what follows, lower case letters denote (real) column vectors or scalars. Matrices are denoted by capital Roman letters. Capital Greek letters, however, denote three-way matrices, and the following definitions are adopted for such an object Ω, with matrices A and B of compatible dimensions, and with repeated indices denoting summation:

$(\Omega')_{ijk} = \Omega_{jki}$ (three-way matrix)

$(A\Omega)_{ijk} = A_{i\sigma}\Omega_{\sigma jk}$

and $(\Omega B)_{ijk} = \Omega_{ij\sigma}B_{\sigma k}$ (three-way matrices)

$\mathrm{tr}(\Omega)_i = \Omega_{\sigma i\sigma}$ (column vector)

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*Mathematician, Research Department. Member AIAA.

Expressions denoting ordinary differential equations with white noise terms should be understood as the formally corresponding stochastic differential equations in the sense of Ito if a rigorous interpretation is desired.

The topic treated here arises from a stochastic optimal control problem with multivariate dynamics of the general form

$$\dot{y} = f(y, v, t, \omega);\ y(t_0) \text{ normal with mean } \bar{y}_0 \text{ a priori}$$

measurements at each time instant t of the form

$$\zeta = g(y, v, t, n)$$

and scalar performance criterion (to be minimized over control laws for v as a function of previous ζ) of the form

$$J = \mathcal{E}\{\psi[y(t_f)] + \int_{t_0}^{t_f} \lambda(y, v, t)\, \mathrm{d}t\}$$

where t_0 and t_f are specified a priori, ω and n are vector white noise "processes," and $\mathcal{E}$ denotes expectation. A related deterministic problem, which is easier to solve, can be defined by making $\omega = 0$, $n = 0$, and $y(t_0) = \bar{y}_0$, and its solution specifies nominal time histories $\bar{y}(t)$, $\bar{v}(t)$, $\bar{\zeta}(t)$, and performance $\bar{J}$. The original problem can then be solved by adding to $\bar{v}$ the perturbation control law for $v - \bar{v}$ that minimizes $J - \bar{J}$. Taylor-series expansions of f, g, ψ, and λ can typically be used to express the dynamics, measurements, and criterion of this optimal perturbation control problem in terms of the noise-induced perturbation variables $y - \bar{y}$, $v - \bar{v}$, $\zeta - \bar{\zeta}$, $J - \bar{J}$, ω, and n. Reference will be made to two asymptotic approximations of such a problem for small perturbations. One is a problem of the standard linear-quadratic-Gaussian form that results from truncating the previous expansions at the lowest nontrivial orders.[1] The other is the more accurate approximation obtained by carrying out these expansions to one higher order, which typically introduces quadratic terms in the dynamics and measurements and cubic terms in the criterion. The units can often be scaled so that the perturbation variables are numerically of order unity and the coefficients of the higher degree terms added in the latter approximation become the quantities that are relatively small, say of order h. Solutions to perturbation control problems that are normalized in this way have been approximated with errors of order h^2 when the other problem parameters and the inverses of two of them (namely, the matrices B and R below) are all of order unity.[2,3] However, R^{-1} is large in many cases of interest because the measurements are highly accurate, and such normalized problems with large R^{-1}, subject to certain conditions, are the topic of this Note.

Class of Perturbation Control Problems

The first of the conditions just mentioned is that the perturbation state can be partitioned as

$$\begin{bmatrix} x \\ \theta \end{bmatrix}$$

such that the linear-quadratic-Gaussian approximation has dynamics of the form

$$\begin{cases} \dot{x} = Fx + Gu + w; & x(t_0) \text{ is normal } (\bar{x}_0, P_0) \text{ a priori} \quad (1) \\ \dot{\theta} = hw_2; & \theta(t_0) \text{ is normal } (0, L_0) \text{ a priori} \quad (2) \end{cases}$$

where u is the perturbation control, perturbation measurements of the form

$$z = Hx + v \tag{3}$$

where z denotes $\zeta - \bar{\zeta}$, and criterion of the form

$$J - \bar{J} = \mathcal{E}[x^T(t_f)S_f x(t_f) + \int_{t_0}^{t_f} (x^T Ax + u^T Bu + 2x^T Zu)\, dt]$$

where (F,G) is stabilizable and (F,H) is observable at each time instant; w, w_2, and v are Gaussian white noise processes with respective covariance matrix parameters Q, D, and R; and $x(t_0)$, $\theta(t_0)$, w, w_2, and v are independent a priori. (This means that θ behaves as a parameter vector in the higher order problem approximation and varies at most very slowly.) Also, A, B, and S_f are symmetric, with A, S_f, P_0, L_0, Q, and D positive semidefinite and B and R positive definite, and in the context of sufficiently small h and an additional problem-formulation parameter m such that $1/h \gg m \gg 1$, all these matrices, $\bar{x}_0$, F, G, H, Z, B^{-1}, and R^{-1}/m^2 are of order unity. F, G, H, B^{-1}, Z, Q, and D may be time varying, but only with time derivatives that are also of order unity, and $\dot{R}^{-1}$ is limited to order m^2.

Another condition is that the derivatives $\partial^2 z_k/\partial\theta_i\, \partial u_j$ can be expressed as $(\Gamma H^T)_{ijk}$ for some three-way matrix $\Gamma(t)$, which is guaranteed if $\dim(z) \le \dim(x)$. Γ is of order h by the nature of the assumed problem normalization.

The remaining conditions are specified in terms of the conditional covariance matrix P of x for Eqs. (1) and (3), which is defined by

$$\dot{P} = FP + PF^T + Q - PH^TR^{-1}HP; \qquad P(t_0) = P_0 \tag{4}$$

for $t_0 \le t \le t_f$, and the related transition matrix $T(s,t)$ defined by

$$\partial T(s,t)/\partial s = [F - PH^TR^{-1}H](s)T(s,t); \qquad T(t,t) = I$$

for $t_0 \le t \le s \le t_f$. These conditions are that a positive $\tau(m)$ exists such that $\lim \tau = 0$ and $\lim m\tau < 1$ as $m \to \infty$, and that for all sufficiently large m and $t \gg \tau$

$$P \text{ is of order } \tau$$

$$\left.\begin{aligned} T(s,t) &\text{ is of order } m\tau e^{(t-s)/\tau} \\ T(s,t)PH^T &\text{ is of order } \frac{e^{(t-s)/\tau}}{m} \end{aligned}\right\} \text{ for } s \ge t$$

and if U and X are symmetric matrix time functions of order unity with X positive semidefinite, if w_3 is Gaussian white noise with covariance parameter X, and if K and q are the matrix time function and vector random process defined by

$$-\dot{K} = K(F - PH^TR^{-1}H) + (F^T - H^TR^{-1}HP)K + U; \quad K(t_f) = 0$$

and

$$\dot{q} = (F - PH^TR^{-1}H)q + mw_3; \qquad q(t_0) = 0$$

then

$$T^T(s,t)KPH^T \text{ is of order } \tau^2 \text{ for } s \ge t$$

and

$$HPKq \text{ is of order } m\tau^{5/2}$$

These last conditions are motivated in particular[3] by the case of an n-dimensional system with $\dot{x}_i = x_{i+1}$, $i = 1, \ldots, n-1$, $\dot{x}_n$ = order-unity noise, and a scalar measurement of x_n only, of which a common example is a second-order system with white-noise acceleration and measurements only of position.[4] These conditions imply that the Kalman filter for Eqs. (1) and (3) can estimate all components of x arbitrarily closely if the measurement noise is small enough, and the conditions might be satisfied in a wide variety of such cases for reasons explained in Ref. 3. This potential filter accuracy is not guaranteed by the observability of (F,H) alone, incidentally; it also depends on the structure of Q. The τ is essentially the "settling time" of such a filter and can be as large as $m^{-1/\dim(x)}$.

Optimal Control Law Properties

For large m with $h^2m^6\tau^{11/2} \ll 1$, a dynamic programming analysis[3] of the higher order approximation of the perturbation control problem shows that the optimal perturbation control law in such a case is asymptotically

$$u = u_0 + B^{-1}\mathrm{tr}\{[\Gamma H^TR^{-1}HPY - (\Lambda G)'']M\} \tag{5}$$

to order $h^2m^2\tau^{5/2}$ when $t \gg \tau$ and $m^2\tau^{5/2} \gg 1$, where u_0 is the order-h approximation of Ref. 2 when $\partial^2 z/\partial\theta\, \partial u$ is replaced by zero, where $Y(t)$ and $\Lambda(t)$ are as specified by

$$-\dot{Y} = Y(F - PH^TR^{-1}H) + (F^T - H^TR^{-1}HP)Y + (SG + Z)B^{-1}(Z^T + G^TS); \qquad Y(t_f) = 0$$

$$-\dot{\Lambda} = \Lambda[F - GB^{-1}(Z^T + G^TS)] + (F^T - H^TR^{-1}HP)\Lambda + YPH^TR^{-1}H\Gamma'B^{-1}(Z^T + G^TS); \qquad \Lambda(t_f) = 0$$

$$-\dot{S} = SF + F^TS + A - (SG + Z)B^{-1}(Z^T + G^TS); \qquad S(t_f) = 0$$

and Eq. (4), and where M is generated from the incoming measurements by

$$\dot{M} = (F - PH^TR^{-1}H)M - (L\Gamma)'B^{-1}(Z^T + G^TS)PH^TR^{-1} \times (z - H\hat{x}); \qquad M(t_0) = 0$$

with $\hat{x}$ and L being the current conditional mean of x and covariance matrix of θ (which are generated to sufficient accuracy in the implementation of u_0). To the accuracy of Eq. (5), the control term added to u_0 is proportional to the state (M) of a high-frequency linear system driven by the innovation $(z - H\hat{x})$ for a Kalman filter with a small settling time. As a result, this added control term fluctuates rapidly and is essentially zero mean, even conditioned on all but very recent data. It therefore has little effect on the system dynamics, and it would seem for this reason that it could only be an example of the probing phenomenon identified by Feldbaum.[5] This phenomenon has been studied extensively[6] but basically refers to the expenditure of current control effort to reduce uncertainty (of θ in this case) to improve future system behavior. However, the interpretation of this control term as probing has not been justified in the manner of Bar-Shalom[6] by an analysis of an explicit decomposition of the performance criterion or of this term's effect on the conditional covariance matrix of x and θ, although the following example provides some evidence that it does indeed reduce the parameter uncertainty. This probing term is of order $h^2m^2\tau^{5/2}$ in general, which can be large compared to h^2 under the assumed conditions when $\dim(x) \ge 2$. It only appears at this order of magnitude when the measurements contain bilinear terms in the parameter and control variables.

Even stronger results can be established for specific cases by using particular properties they have beyond those assumed above. In the type of second-order system mentioned previously, for example, Eq. (5) can be valid even when the probing term is larger than h, making it the most significant term in the optimal perturbation control after those derived from the linear-quadratic-Gaussian approximation of the problem.[4] Figure 1 shows the effect that this probing can have in such a

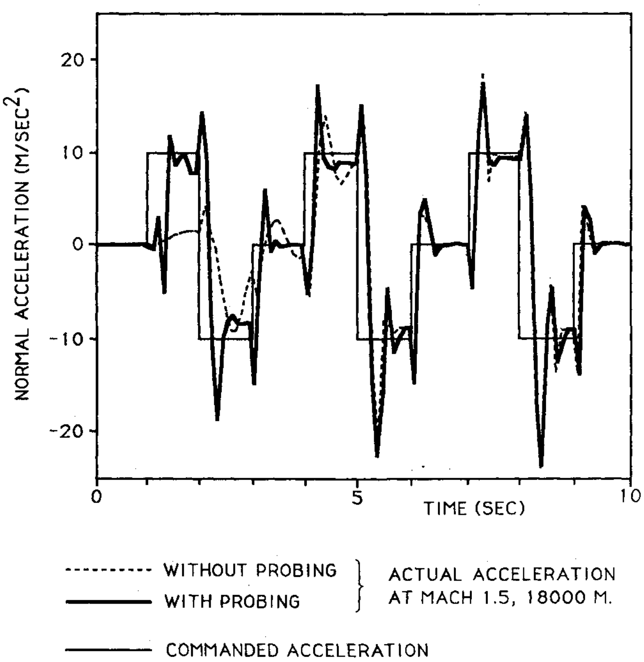

Fig. 1 Autopilot response in a realistic computer simulation of missile and aerodynamics.

case when implemented judiciously. This example is an adaptive missile autopilot[3,4] where the controller measures the missile's acceleration normal to the flight path and seeks to make it follow a commanded acceleration by rotating the missile with tail-fin deflections (the control variable) to create body lift. Instead of being limited by established theory, the prior variances of the (two) unnormalized parameters, which are proportional to h^2 and reflect variability in the flight conditions, were made as large as would empirically allow reasonable behavior at the nominal flight condition, which is Mach 2.0 at 6000 m altitude. The probing control term has little effect then but substantially improves the initial performance for the very different flight condition of Fig. 1 by causing the parameter estimates to converge more quickly to the corresponding nonzero values, which occurs because this control term reduces the uncertainties of these parameters more quickly.

Conclusions

Asymptotic approximations of optimal control laws have been established for a class of dynamic systems whose state components are either slowly varying (i.e., parameters) or can be tracked arbitrarily well by making the controller's measurements sufficiently accurate. These control optimization problems have the form assumed by higher order descriptions of noise-induced perturbations from optimal nominal behavior in a more general type of control problem. When the measurement noise is small, bilinear measurement terms in the parameter and control variables can give rise to a rapidly fluctuating "probing" term in the optimal control. This term is the output of a high-frequency linear system driven by a product of the parameter covariance matrix and a Kalman filter innovation vector.

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Deployment of a Flexible Beam from an Oscillating Base

Nelson G. Creamer*
Swales and Associates, Inc.,
Beltsville, Maryland 20705

Introduction

MANY past, present, and future spacecraft employ long flexible appendages that must be launch-stored for compactness and subsequently deployed in orbit. An extensive amount of research and analysis has been directed toward the dynamics associated with deployment of flexible appendages from both a fixed base and a host spacecraft (see, for example, Refs. 1–5). To this date, however, very little analysis has focused on the dynamic interaction of a flexible appendage and its deployment mechanism, which, as observed during deployment and retraction of the Shuttle-based solar array flight experiment (SAFE), may result in large flexural deformations and possible damage to the deploying boom.[6] These structural deformations are attributed to a resonant interaction of the instantaneous appendage natural frequencies and a slight oscillatory motion within the deployment mechanism. The objective of this Note is to examine this resonant interaction using Timoshenko beam theory in conjunction with base oscillatory motion.

Governing Equations

The system of interest is a flexible beam with a tip payload that is axially deployed from a fixed canister, as shown in Fig. 1. Internal to the canister is a motor-driven rotating system that uncoils and deploys the appendage. This rotating mechanism produces a small periodic translational motion [$\delta(t)$] and a small periodic rotational motion [$\gamma(t)$] of the base of the appendage. An internal view of a typical canister, similar to the ones used on the SAFE experiment and the Naval Research Laboratory's Low-power Atmospheric Compensation Experiment (LACE), is shown in the enlargement of Fig. 1. It is observed that there are three regions of interest: 1) a storage region where a truss structure is coiled and stored; 2) a turntable region that rotates at a prescribed frequency, thereby uncoiling the truss; and 3) an elevating region in which nuts and rails are used to guide and deploy the truss as it uncoils. It is in this third region that the base motion can occur due to an interaction of the rotating, uncoiling truss structure with slight freeplay in the roller system.

Lagrange's equations are used to derive the equations of motion for the system. The potential energy can be written in the form

$$V = \frac{1}{2}\int_0^L \left[EI\left(\frac{\partial \alpha}{\partial x}\right)^2 + \kappa GA\left(\alpha - \frac{\partial v}{\partial x}\right)^2 \right] \mathrm{d}x \qquad (1)$$

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*Engineer. Member AIAA.